TRANSPORT EQUATIONS, FINITE-VOLUME SCHEMES AND COUPLING ALGORITHMS

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1. Transport Equations and Finite-Volume Schemes

Transport in the core of a tokamak is approximately described by a system of one-dimensional equations for the evolution of density, and energy for each species in the plasma and a net momentum equation. In addition, the magnetic field geometry is also evolved. For most applications only ion densities, electron and ion energies are evolved.

The key challenge in one-dimensional transport modelling is determination of accurate flux models. Over the years many complex and sophisticated flux models, of varying physical fidelity, have been developed. Even though the physical realism of core transport calculations has increased over the years, ultimately core transport remains an approximate description and recourse to more fundamental descriptions must be made to fully understand and describe the complex physics in a tokamak.

The system of transport equations can be written in the form

$$\frac{\partial \mathbf{Q}}{\partial t} + \frac{1}{V'}\frac{\partial}{\partial x}\left(V'\mathbf{\Gamma}\right) = \mathbf{S} \tag{1}$$

where $\mathbf{Q} = \mathbf{Q}(x,t)$ is a m-dimensional vector of conserved quantities (density, momentum, temperature), $\mathbf{\Gamma} = \mathbf{\Gamma}(\mathbf{Q}, \partial\mathbf{Q}/\partial x)$ is a flux function, $\mathbf{S} = \mathbf{S}(\mathbf{Q}, x, t)$ is a source function. The coordinate x is dimensionless and represents normalized distance from the magnetic axis, $x = 0$, to the separatrix, $x = 1$. The equation is solved on the domain $[0, x_e]$, where $x_e \le 1$. The function $V(x)$ represents the volume of plasma (in m^3) enclosed between $[0, x]$ and encodes the geometry of the tokamak core region. Note that $V'(x)$ is related to the metric coefficient that appears in a divergence operator written in generalized coordinate.

The point $x = 0$ represents a symmetry axis in the tokamak which means we must set $\mathbf{\Gamma} = 0$ at $x = 0$. At $x = x_e$ one can either specify the flux, $\mathbf{\Gamma}(x_e) = \mathbf{\Gamma}_e(t)$, or the values of the conserved variables $\mathbf{Q}(x_e, t) = \mathbf{Q}_e(t)$, where $\mathbf{\Gamma}_e(t)$ and $\mathbf{Q}_e(t)$ are specified functions.

To discretize the system of equations consider a finite-volume grid in which a cell is defined as $C_i \equiv [x_i, x_{i+1}]$, with $i = 1, \ldots, N$. Multiplying the transport equation by $V'(x)$ and integrating over a cell and using finite-differences for the time derivative we get the update formula

$$\mathbf{Q}_i^{n+1} = \mathbf{Q}_i^n - \frac{\Delta t}{V'(x_i)\Delta x_i}\left(V'(x_{i+1/2})\mathbf{F}_{i+1/2}^{n+\theta} - V'(x_{i-1/2})\mathbf{F}_{i-1/2}^{n+\theta}\right) + \Delta t \mathbf{S}_i^{n+\theta} \tag{2}$$

where $0 \le \theta \le 1$, $\mathbf{S}_i^{n+\theta} = \mathbf{S}(\mathbf{Q}_i^{n+\theta}, x, t)$ and

$$\mathbf{F}_{i-1/2}^{n+\theta} = \mathbf{H}(\mathbf{Q}_{i-1}^{n+\theta}, \mathbf{Q}_i^{n+\theta}) \tag{3}$$

is a numerical flux function that represents the flux at the cell edge. Here, $x_{i+1/2} \equiv (x_i + x_{i+1})/2$ and $\Delta x_i \equiv x_{i+1} - x_i$. Using a Taylor series expansion we can show that the update formula has $O(\Delta x^2)$ spatial accuracy. Different temporal accuracy can be achieved by choosing θ: for example, $\theta = 0$ or $\theta = 1$ will give $O(\Delta t)$ accuracy while $\theta = 1/2$ will give $O(\Delta t^2)$ accuracy. Values of $\theta > 0$ lead to an implicit scheme as $Q^{n+\theta}$ is then not known. The stability properties of the scheme will also depend on θ. To compute $Q^{n+\theta}$ one can use, for example, a linear combination of old and new values, i.e. $(1-\theta)Q^n + \theta Q^{n+1}$. A predictor-corrector method can also be uses, however, this will generally involve extra flux evaluations that can be expensive in some transport calculations.

The numerical flux function must satisfy the consistency condition

$$\lim_{\epsilon \to 0} \mathbf{H}\left(\mathbf{Q}(x-\epsilon, t), \mathbf{Q}(x+\epsilon, t)\right) = \mathbf{\Gamma}(\mathbf{Q}, \partial \mathbf{Q}/\partial x) \tag{4}$$

which ensures the convergence of the solution as $\Delta x \to 0$ and $\Delta t \to 0$.

The boundary condition at $x = 0$ can be applied by simply setting $\mathbf{F}_{1/2} = 0$. If the flux is specified at $x = x_e$, then this can be enforced using $\mathbf{F}_{N+1/2} = \mathbf{F}_e(t)$. For the second type of boundary conditions (specified conserved variables) we introduce a "ghost-cell" the same size as the last cell C_N. Then, if we set $\mathbf{Q}_{N+1} = 2\mathbf{Q}_e(t) - \mathbf{Q}_N$, we will enforce $\mathbf{Q}(x_e, t) = \mathbf{Q}_e(t)$ to O(Δx^2). Using this in the definition of the numerical flux, we get the flux at the last edge as

$$\mathbf{F}_{N+1/2}^{n+\theta} = \mathbf{H}(\mathbf{Q}_N^{n+\theta}, 2\mathbf{Q}_e(t+\theta\Delta t) - \mathbf{Q}_N^{n+\theta}). \tag{5}$$

2. Coupling schemes

To explore *coupling schemes* we can imagine solving Eq. (1) separately on $x < x_c$ and $x > x_c$ and then imposing additional conditions at $x = x_c$. These *coupling conditions* are continuity of the solution and the fluxes as shown from the following lemma

Lemma 1. *Let $0 < x_c < x_e$ be some point in $[0, x_e]$. Then $\mathbf{Q}(x,t)$ is a solution to Eq. (1) in $[0, x_e]$ if it a smooth solution in $x < x_c$ and $x > x_c$ and satisfies the conditions*

$$\mathbf{Q}_- = \mathbf{Q}_+ \tag{6}$$

$$\mathbf{\Gamma}_- = \mathbf{\Gamma}_+ \tag{7}$$

where $\mathbf{Q}_{\pm} \equiv \lim_{\epsilon \to 0} \mathbf{Q}(x_c \pm \epsilon, t)$ *and* $\mathbf{\Gamma}_{\pm} \equiv \mathbf{\Gamma}(\mathbf{Q}_{\pm}, \partial \mathbf{Q}_{\pm}/\partial x)$.