THE EIGENSYSTEM OF THE MAXWELL EQUATIONS WITH EXTENSION TO PERFECTLY HYPERBOLIC MAXWELL EQUATIONS.

AMMAR H. HAKIM

1. Eigensystem of Maxwell equations

In this document I list the eigensystem of the Maxwell equations. Maxwell's equations consist of the curl equations

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} = 0 \tag{1}$$

$$\epsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t} - \nabla \times \mathbf{B} = -\mu_0 \mathbf{J} \tag{2}$$

along with the divergence relations

$$\nabla \cdot \mathbf{E} = \frac{\varrho_c}{\epsilon_0} \tag{3}$$

$$\nabla \cdot \mathbf{B} = 0. \tag{4}$$

Here, $\mathbf{E}$ is the electric field, $\mathbf{B}$ is the magnetic flux density, ϵ_0, μ_0 are permittivity and permeability of free space, and $\mathbf{J}$ and ϱ are specified currents and charges respectively. The speed of light is determined from $c = 1/(\mu_0 \epsilon_0)^{1/2}$.

These are linear equations and hence the eigensytem is independent of the value of the electromagnetic fields. In 1D Maxwell equations can be written as, ignoring sources,

$$\frac{\partial}{\partial t} \begin{bmatrix} E_x \\ E_y \\ E_z \\ B_x \\ B_y \\ B_z \end{bmatrix} + \frac{\partial}{\partial x} \begin{bmatrix} 0 \\ c^2 B_z \\ -c^2 B_y \\ 0 \\ -E_z \\ E_y \end{bmatrix} = 0. \tag{5}$$

The eigenvalues of this system are $\{0, 0, c, c, -c, -c\}$. The right eigenvectors of the flux Jacobian are given by the columns of the matrix

$$R = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & c & 0 & -c & 0 \\ 0 & 0 & 0 & -c & 0 & c \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}. \tag{6}$$

The left eigenvectors are the rows of the matrix

$$L = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2c} & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & -\frac{1}{2c} & 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{2c} & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2c} & 0 & \frac{1}{2} & 0 \end{bmatrix}. \tag{7}$$

2. Eigensystem of Perfectly Hyperbolic Maxwell equations

The perfectly hyperbolic Maxwell equations are a modification of the Maxwell equations that take into account the divergence relations. The modified equations explicitly "clean" divergence errors and are a hyperbolic generalization of the Hodge project method commonly used in electromagnetism to correct for charge conservation errors[2, 1, 3].

These equations are written as Eq. (3) and Eq. (4).

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} + \gamma \nabla \psi = 0 \tag{8}$$

$$\epsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t} - \nabla \times \mathbf{B} + \chi \nabla \phi = -\mu_0 \mathbf{J} \tag{9}$$

$$\frac{1}{\chi} \frac{\partial \phi}{\partial t} + \nabla \cdot \mathbf{E} = \frac{\varrho_c}{\epsilon_0} \tag{10}$$

$$\frac{\epsilon_0 \mu_0}{\gamma} \frac{\partial \psi}{\partial t} + \nabla \cdot \mathbf{B} = 0. \tag{11}$$

Here, ψ and ψ are correction potentials for the electric and magnetic field respectively and χ and γ are dimensionless factors that control the speed at which the errors are propagated.

In 1D these equations can be written as, ignoring sources,

$$\frac{\partial}{\partial t} \begin{bmatrix} E_x \\ E_y \\ E_z \\ B_x \\ B_y \\ B_z \\ \phi \\ \psi \end{bmatrix} + \frac{\partial}{\partial x} \begin{bmatrix} \chi c^2 \phi \\ c^2 B_z \\ -c^2 B_y \\ \gamma \psi \\ -E_z \\ E_y \\ \chi E_x \\ \gamma c^2 B_x \end{bmatrix} = 0. \tag{12}$$

The eigenvalues of this system are $\{-c\gamma, c\gamma, -c\chi, c\chi, c, c, -c, -c\}$. The right eigenvectors of the flux Jacobian are given by the columns of the matrix

$$R = \begin{bmatrix} 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c & 0 & -c & 0 \\ 0 & 0 & 0 & 0 & 0 & -c & 0 & c \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{c} & \frac{1}{c} & 0 & 0 & 0 & 0 \\ -c & c & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{13}$$

The left eigenvectors are the rows of the matrix

$$L = \begin{bmatrix} 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & -\frac{1}{2c} \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2c} \\ \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & -\frac{c}{2} & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & \frac{c}{2} & 0 \\ 0 & \frac{1}{2c} & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{1}{2c} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & -\frac{1}{2c} & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2c} & 0 & \frac{1}{2} & 0 & 0 & 0 \end{bmatrix}. \tag{14}$$

References

[1] C.-D Munz, P. Omnes, and R. Schneider. A three-dimensional finite-volume solver for the Maxwell equations with divergence cleaning on unstructured meshes. *Computer Physics Communications*, 130:83–117, 2000.

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[3] C.-D Munz and U. Voß. A finite-volume method for the Maxwell equations in the time domain. *SIAM Journal of Scientific Computing*, 22(2):449–475, 2000.