THE EIGENSYSTEM OF THE EULER EQUATIONS

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In this document I list the eigensystem of the Euler equations. The formulas are taken from[1], Chapter 3, section 3.1. The Euler equations can be written in conservative form as

$$\frac{\partial}{\partial t}\begin{bmatrix}\rho\\ \rho u\\ \rho v\\ \rho w\\ E\end{bmatrix}+\frac{\partial}{\partial x}\begin{bmatrix}\rho u\\ \rho u^2+p\\ \rho uv\\ \rho uw\\ (E+p)u\end{bmatrix}=0 \tag{1}$$

where

$$E=\rho\varepsilon+\frac{1}{2}\rho q^2 \tag{2}$$

is the total energy and ε is the internal energy of the fluid and $q^2=u^2+v^2+w^2$. The pressure is given by an equation of state (EOS) $p=p(\varepsilon,\rho)$. For an ideal gas the EOS is $p=(\gamma-1)\rho\varepsilon$.

The eigenvalues are $\{u-c,u,u,u,u+c\}$. The right eigenvectors of the flux Jacobian are given by

$$R=\begin{bmatrix}1 & 0 & 0 & 1 & 1\\ u-c & 0 & 0 & u & u+c\\ v & 1 & 0 & v & v\\ w & 0 & 1 & w & w\\ h-uc & v & w & h-c^2/b & h+uc\end{bmatrix} \tag{3}$$

here

$$h=(E+p)/\rho \tag{4}$$

$$c=\sqrt{\frac{\partial p}{\partial\rho}+\frac{p}{\rho^2}\frac{\partial p}{\partial\varepsilon}} \tag{5}$$

is the enthalpy and the sound speed respectively. Also,

$$b=\frac{1}{\rho}\frac{\partial p}{\partial\varepsilon}. \tag{6}$$

Note that for ideal gas EOS we have

$$h=\frac{c^2}{\gamma-1}+\frac{1}{2}q^2 \tag{7}$$

$$c=\sqrt{\frac{\gamma p}{\rho}} \tag{8}$$

and $b = \gamma - 1$. Hence, in this case the term $h - c^2/b$ in Eq. (9) is just $q^2/2$. The left eigenvectors are

$$L = \frac{b}{2c^2}\begin{bmatrix} \theta + uc/b & -u - c/b & -v & -w & 1 \\ -2vc^2/b & 0 & 2c^2/b & 0 & 0 \\ -2wc^2/b & 0 & 0 & 2c^2/b & 0 \\ 2h - 2q^2 & 2u & 2v & 2w & -2 \\ \theta - uc/b & -u + c/b & -v & -w & 1 \end{bmatrix} \tag{9}$$

where

$$\theta = q^2 - \frac{E}{\rho} + \rho\frac{\partial p/\partial \rho}{\partial p/\partial \varepsilon} \tag{10}$$

which, for an ideal gas EOS reduces to $q^2/2$.

Now consider the problem of splitting a jump vector $\Delta \equiv [\delta_0, \delta_1, \delta_2, \delta_3, \delta_4]^T$ into coefficients neeeded in computing the Riemann problem. The coefficients are given by $L\Delta$. For an ideal gas law EOS, after some algebra we can show that an efficient way to compute these are

$$\alpha_3 = \frac{\gamma - 1}{c^2}\left[(h - q^2)\delta_0 + u\delta_1 + v\delta_2 + w\delta_3 - \delta_4\right] \tag{11}$$

$$\alpha_1 = -v\delta_0 + \delta_2 \tag{12}$$

$$\alpha_2 = -w\delta_0 + \delta_3 \tag{13}$$

$$\alpha_4 = \frac{1}{2c}\left[\delta_1 + (c - u)\delta_0 - c\alpha_3\right] \tag{14}$$

$$\alpha_0 = \delta_0 - \alpha_3 - \alpha_4. \tag{15}$$

References

[1] Andrei G. Kulikoviskii, Nikolai V. Pogorelov, and Andrei Yu. Semenov. *Mathematical Aspects of Numerical Solutions of Hyperbolic Systems*. Chapman and Hall/CRC, 2001.