MULTIFLUID FLOWING EQUILIBRIUM EQUATIONS

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1. Governing Equations

In this document I derive and list equations governing axisymmetric multifluid equilibria. This derivation is largely based on the paper by Steinhauer and Ishida[1] except that I do not normalize the equations. Also, I do not pursue the "nearby fluids" concept introduced in that paper.

The basic governing equations are the steady-state two-fluid equations in which the electron mass is set to zero. For each fluid the continuity and pressure equations are

$$\nabla \cdot (n_\alpha \mathbf{u}_\alpha) = 0 \tag{1}$$

$$\mathbf{u}_\alpha \cdot \nabla p_\alpha = -\gamma p_\alpha \nabla \cdot \mathbf{u}_\alpha \tag{2}$$

For each ion species the momentum equation is

$$\mathbf{u}_\alpha \cdot \nabla \mathbf{u}_\alpha = -\frac{\nabla p_\alpha}{m_\alpha n_\alpha} + \frac{q_\alpha}{m_\alpha}(\mathbf{E} + \mathbf{u}_\alpha \times \mathbf{B}) \tag{3}$$

while for the electrons, the momentum equation reduces to

$$0 = -\nabla p_e - e n_e (\mathbf{E} + \mathbf{u}_e \times \mathbf{B}). \tag{4}$$

In these equations m_α and q_α are the species charge and mass respectively, n_α is the number density, $\mathbf{u}_\alpha$ the velocity and p_α the pressure. For smooth flows the pressure equation can be replaced by an advection equation for the entropy that is obtained by setting $p_\alpha = n_\alpha^\gamma e^{(\gamma-1)s_\alpha}$ to give

$$\mathbf{u}_\alpha \cdot \nabla s_\alpha = 0. \tag{5}$$

The electromagnetic field is determined from the steady-state Maxwell equations

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \tag{6}$$

$$\nabla \times \mathbf{E} = \mathbf{0} \tag{7}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{8}$$

where $\mathbf{J} = \sum_\alpha q_\alpha n_\alpha \mathbf{u}_\alpha$ is the total plasma current. Finally, the condition of quasi-neutrality is used to compute the electron density.

2. Flux Functions

As the divergence of the fluid momentum and magnetic field vanishes (see Eq. (1) and Eq. (8)), in axisymmetric geometry we can write, using identities (39) and (43),

$$\mathbf{u}_\alpha = u_{\alpha\phi}\mathbf{e}_\phi + \frac{1}{rn_\alpha}\nabla\psi_\alpha \times \mathbf{e}_\phi \tag{9}$$

$$\mathbf{B} = B_\phi\mathbf{e}_\phi + \frac{1}{r}\nabla\psi \times \mathbf{e}_\phi \tag{10}$$

where $u_{\alpha\phi}$ and B_ϕ are the toroidal fluid velocity and magnetic field respectively and $\psi_\alpha(r,z)$ and $\psi(r,z)$ are scalar flux functions that determine the poloidal fluid velocity and magnetic fields. The total plasma current can be hence expressed as

$$\mathbf{J} = \sum_\alpha q_\alpha n_\alpha u_{\alpha\phi} + \frac{1}{r}\sum_\alpha q_\alpha \nabla\psi_\alpha \times \mathbf{e}_\phi \tag{11}$$

where the summation is over the electrons and all ion species in the plasma. Using Eq. (6) and the identity (41) in this equation for the current we get

$$rB_\phi = \mu_0 \sum_\alpha q_\alpha\psi_\alpha \tag{12}$$

$$-\frac{\triangle^*\psi}{r} = \mu_0 \sum_\alpha q_\alpha n_\alpha u_{\alpha\phi}. \tag{13}$$

These first of these equations relates the toroidal magnetic field to the plasma flux functions while the second one relates the magnetic field flux function to the total toroidal current. Alternately, the second equation can be rewritten as an equation for the toroidal electron current in terms of the ion toroidal currents and the magnetic field flux function.

To simplify the fluid momentum equations we introduce the canonical momentum defined by

$$\mathbf{P}_\alpha = m_\alpha\mathbf{u}_\alpha + q_\alpha\mathbf{A} \tag{14}$$

where $\mathbf{A}$ is the vector potential in terms of which $\mathbf{B} = \nabla\times\mathbf{A}$. We also define the canonical vorticity as $\boldsymbol{\Omega}_\alpha = \nabla\times\mathbf{P}_\alpha = m_\alpha\boldsymbol{\omega}_\alpha + q_\alpha\mathbf{B}$ where $\boldsymbol{\omega}_\alpha = \nabla\times\mathbf{u}_\alpha$ is the fluid vorticity. As $\nabla\cdot\boldsymbol{\Omega}_\alpha = 0$ we can write

$$\boldsymbol{\Omega}_\alpha = \Omega_{\alpha\phi}\mathbf{e}_\phi + \frac{1}{r}\nabla Y_\alpha \times \mathbf{e}_\phi \tag{15}$$

where $Y_\alpha(r,z)$ is a canonical vorticity flux function. Using the definition of canonical vorticity and the identity (44) to express $\boldsymbol{\omega}_\alpha$ and using Eq. (10) to express the magnetic field we get, comparing to Eq. (15),

$$\Omega_{\alpha\phi} = -\frac{m_\alpha}{r}\triangle^*_{n_\alpha}\psi_\alpha + q_\alpha B_\phi \tag{16}$$

$$Y_\alpha = m_\alpha r u_{\alpha\phi} + q_\alpha\psi. \tag{17}$$

We next use the identity

$$\mathbf{u}_\alpha \cdot \nabla \mathbf{u}_\alpha + \mathbf{u}_\alpha \times \nabla \times \mathbf{u}_\alpha = \nabla(\mathbf{u}_\alpha^2/2) \tag{18}$$

and the thermodynamic relation

$$\frac{\nabla p_\alpha}{n_\alpha} = \nabla h_\alpha - T_\alpha \nabla s_\alpha \tag{19}$$

where $h_\alpha = \gamma T/(\gamma - 1)$ is the fluid enthalpy[1] and T_α is the fluid temperature (defined as $p_\alpha = n_\alpha T_\alpha$) in the momentum equation we get

$$\nabla \left(h_\alpha + m_\alpha \mathbf{u}_\alpha^2/2 + q_\alpha \phi \right) = T_\alpha \nabla s_\alpha + \mathbf{u}_\alpha \times \mathbf{\Omega}_\alpha. \tag{20}$$

This equation is only valid for the ions. For the electrons we need to set $m_e = 0$ and $q_e = -e$ to get the simplified electron momentum equation as

$$\nabla \left(h_e - e\phi \right) = T_e \nabla s_e - e \mathbf{u}_e \times \mathbf{B} \tag{21}$$

where we have use the fact that for electrons the canonical vorticity is simply $\mathbf{\Omega}_e = -e\mathbf{B}$.

3. Surface Functions. Components of Momentum Equations

To derive the final set of equations governing the multifluid equilibrium we need to look at the components of the ion and electron momentum equations. First, we look at the toroidal component by taking the dot product with $\mathbf{e}_\phi$. As the toroidal component (in axisymmetric geometry) of the gradient of a scalar vanishes, we get the conditions

$$0 = (\mathbf{u}_\alpha \times \mathbf{\Omega}_\alpha) \cdot \mathbf{e}_\phi \tag{22}$$

$$0 = (\mathbf{u}_e \times \mathbf{B}) \cdot \mathbf{e}_\phi. \tag{23}$$

Using the identity (47) we can show that these reduce to

$$0 = (\nabla \psi_\alpha \times \nabla Y_\alpha) \cdot \mathbf{e}_\phi \tag{24}$$

$$0 = (\nabla \psi_e \times \nabla \psi) \cdot \mathbf{e}_\phi. \tag{25}$$

This shows[2] that the flux functions ψ_α and ψ_e are *surface* functions, i.e.

$$\psi_\alpha = \overline{\psi}_\alpha(Y_\alpha) \tag{26}$$

$$\psi_e = \overline{\psi}_e(\psi). \tag{27}$$

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[1] The usual definition of enthalpy for an ideal fluid is $h = \gamma T/(\gamma - 1) + m\mathbf{u}^2/2$. In the definition adopted here the kinetic energy contribution is left out but taken into account (for the ions) in the momentum equation.

[2] Whenever we have scalar functions that are related by $\nabla\psi(r,z) = K(r,z)\nabla\phi(r,z)$ we can show $\psi(r,z) = \overline{\psi}(\phi(r,z))$ i.e., the function ψ is can be written as a *surface* function of the scalar field ϕ instead of (r,z). This also implies that $\nabla\psi = \overline{\psi}'\nabla\phi$, where the prime denotes differentiation with respect to ϕ.

With these expressions we can write the poloidal component of the ion velocity (see Eq. (9)) as

$$\mathbf{u}_{\alpha p} = \frac{1}{rn_\alpha}\nabla\psi_\alpha \times \mathbf{e}_\phi = \frac{\overline{\psi}'_\alpha}{rn_\alpha}\nabla Y_\alpha \times \mathbf{e}_\phi = \frac{\overline{\psi}'_\alpha}{n_\alpha}\mathbf{\Omega}_{\alpha p} \tag{28}$$

where $\mathbf{\Omega}_{\alpha p}$ is the poloidal component of the canonical vorticity. Similarly, the poloidal component of the electron velocity is

$$\mathbf{u}_{ep} = \frac{1}{rn_e}\nabla\psi_e \times \mathbf{e}_\phi = \frac{\overline{\psi}'_e}{rn_e}\nabla\psi \times \mathbf{e}_\phi = \frac{\overline{\psi}'_e}{n_e}\mathbf{B}_p \tag{29}$$

where $\mathbf{B}_p$ is poloidal magnetic field. Hence, the ion and electron poloidal flows are not, in general, parallel to each other.

We can show that the fluid entropy are surface functions by using identity (48) in Eq. (5) to get

$$\frac{1}{rn_\alpha}(\nabla s_\alpha \times \nabla\psi_\alpha)\cdot\mathbf{e}_\phi = 0 \tag{30}$$

which allows us to write $s_\alpha = \overline{s}_\alpha(Y_\alpha)$ for the ions and $s_e = \overline{s}_e(\psi)$ for the electrons.

We now take the component of the ion and electron momentum equations along $\mathbf{u}_\alpha$. The right hand side vanishes (see Eq. (5)) which yields, upon using identity (48)

$$h_\alpha + \frac{1}{2}m_\alpha\mathbf{u}_\alpha^2 + q_\alpha\phi = \overline{H}_\alpha(Y_\alpha) \tag{31}$$

for the ions and

$$h_e - e\phi = \overline{H}_e(\psi) \tag{32}$$

for the electrons. These equations are a form of Bernoulli's equations and state that the total enthalpy (including the electrostatic potential energy) of each fluid is constant on a flux surface.

Finally, we take the component of the ion and electron momentum equations along ∇Y_α and $\nabla\psi$ respectively. These directions are perpendicular to the poloidal components of the ion and electron velocities and hence will yield independent set of equations. For ions taking the dot product with ∇Y_α we get

$$\overline{H}'_\alpha\nabla Y_\alpha\cdot\nabla Y_\alpha = T_\alpha\overline{s}'_\alpha\nabla Y_\alpha\cdot\nabla Y_\alpha + \nabla Y_\alpha\cdot(\mathbf{u}_\alpha\times\mathbf{\Omega}_\alpha). \tag{33}$$

The last term in this equation can be simplified using the identity (47) to get, after some rearrangements, the differential equation

$$m_\alpha\overline{\psi}'_\alpha r^2\nabla\cdot\left(\frac{\overline{\psi}'_\alpha}{r^2 n_\alpha}\nabla Y_\alpha\right) = r\big(\overline{\psi}'_\alpha q_\alpha B_\phi - n_\alpha u_{\alpha\phi}\big) + n_\alpha r^2\big(\overline{H}'_\alpha - T_\alpha\overline{s}'_\alpha\big) \tag{34}$$

Here, we have rewritten the density-weighted Grad-Shafranov operator as

$$r^2\nabla\cdot\left(\frac{1}{r^2 n_\alpha}\nabla\psi_\alpha\right) = r^2\nabla\cdot\left(\frac{\overline{\psi}'_\alpha}{r^2 n_\alpha}\nabla Y_\alpha\right) \tag{35}$$

to make the ion flux function Y_α the independent variable. For electrons taking the dot product with $\nabla\psi$ we get

$$\overline{H}'_e\nabla\psi\cdot\nabla\psi = T_e\overline{s}'_e\nabla\psi\cdot\nabla\psi - \nabla\psi\cdot(e\mathbf{u}_e\times\mathbf{B}). \tag{36}$$

As we did for the ions, the last term in this equation can be simplified using the identity (47) to get, after some rearrangements, the equation

$$ren_e u_{e\phi} = rJ_{e\phi} = reB_\phi\overline{\psi}'_e - n_e r^2\big(\overline{H}'_e - T_e\overline{s}'_e\big) \tag{37}$$

Note that the left hand side of this equation involves $J_{e\phi}$, i.e., the toroidal component of the electron current. This can be eliminated from Eq. (13) to get a differential equation for the magnetic field flux function $\psi(r,z)$

$$\triangle^*\psi = \mu_0 r\big(eB_\phi\overline{\psi}'_e - J_{i\phi}\big) - \mu_0 n_e r^2\big(\overline{H}'_e - T_e\overline{s}'_e\big) \tag{38}$$

where $J_{i\phi} \equiv \sum_\alpha q_\alpha n_\alpha u_{\alpha\phi}$ and the sum is taken over all ion species[3].

4. Summary

We have derived a set of equations that describes multifluid equilibrium configurations. The electron surface functions are $\overline{\psi}_e(\psi)$, $\overline{s}_e(\psi)$ and $\overline{H}_e(\psi)$. For each ion species we have the same set of surface functions but these are now functions of a *different* flux function, Y_α: $\overline{\psi}_\alpha(Y_\alpha)$, $\overline{s}_\alpha(Y_\alpha)$ and $\overline{H}_\alpha(Y_\alpha)$. These functions are *arbitrary* and need to be specified before the equations can be solved.

Appendix A. Useful Identities

Let $\mathbf{a}$ be an axisymmetric vector field satisfying $\nabla\cdot\mathbf{a} = 0$. Then, in cylindrical coordinates, it can be written as

$$\mathbf{a} = a_\phi\mathbf{e}_\phi + \frac{1}{r}\nabla\psi\times\mathbf{e}_\phi, \tag{39}$$

where $\mathbf{e}_\phi$ are unit vectors and $\psi = \psi(r,z)$ is an arbitrary function. In component form

$$a_r = -\frac{1}{r}\frac{\partial\psi}{\partial z}, \quad a_z = \frac{1}{r}\frac{\partial\psi}{\partial r}. \tag{40}$$

The curl of $\mathbf{a}$ is given by

$$\nabla\times\mathbf{a} = -\frac{\triangle^*\psi}{r}\mathbf{e}_\phi + \frac{1}{r}\nabla(ra_\phi)\times\mathbf{e}_\phi, \tag{41}$$

[3] Comparing the ion and electron equation we see a difference: the total ion *current* appears in the electron equation while the ion *velocity* appears in the ion equation. The reason for this is that the units of ψ and ψ_α are not the same.

where $\triangle^*$ is the *Grad-Shafranov* operator defined by

$$\triangle^*\psi \equiv \frac{\partial^2\psi}{\partial z^2} + r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\right) = r^2\nabla\cdot\left(\frac{1}{r^2}\nabla\psi\right) \tag{42}$$

If $\mathbf{a}$ is a axisymmetric vector field and $f(r,z)$ is a scalar function and $\nabla\cdot(f\mathbf{a}) = 0$, then

$$\mathbf{a} = a_\phi\mathbf{e}_\phi + \frac{1}{rf}\nabla\psi\times\mathbf{e}_\phi. \tag{43}$$

The curl of $\mathbf{a}$ is given by

$$\nabla\times\mathbf{a} = -\frac{\triangle^*_f\psi}{r}\mathbf{e}_\phi + \frac{1}{r}\nabla(ra_\phi)\times\mathbf{e}_\phi, \tag{44}$$

where $\triangle^*_f$ is a *f-weighted Grad-Shafranov* operator defined by

$$\triangle^*_f\psi \equiv \frac{\partial}{\partial z}\left(\frac{1}{f}\frac{\partial\psi}{\partial z}\right) + r\frac{\partial}{\partial r}\left(\frac{1}{rf}\frac{\partial\psi}{\partial r}\right) = r^2\nabla\cdot\left(\frac{1}{r^2 f}\nabla\psi\right) \tag{45}$$

Let $\mathbf{a} = a_\phi\mathbf{e}_\phi + \nabla\psi_a\times\mathbf{e}_\phi/f_a r$ and $\mathbf{b} = b_\phi\mathbf{e}_\phi + \nabla\psi_b\times\mathbf{e}_\phi/f_b r$ where $f_a = f_a(r,z)$ and $f_b = f_b(r,z)$ are scalar fields. Then

$$\mathbf{a}\times\mathbf{b} = \frac{a_\phi}{rf_b}\nabla\psi_b - \frac{b_\phi}{rf_a}\nabla\psi_a - \frac{1}{r^2 f_a f_b}(\nabla\psi_a\times\mathbf{e}_\phi\cdot\nabla\psi_b)\mathbf{e}_\phi \tag{46}$$

$$= \frac{a_\phi}{rf_b}\nabla\psi_b - \frac{b_\phi}{rf_a}\nabla\psi_a + \frac{1}{r^2 f_a f_b}\nabla\psi_a\times\nabla\psi_b. \tag{47}$$

Let $\mathbf{a} = a_\phi\mathbf{e}_\phi + \nabla\psi\times\mathbf{e}_\phi/r$ and $f = f(r,z)$ is a scalar field. Then

$$\mathbf{a}\cdot\nabla f = \frac{1}{r}(\nabla f\times\nabla\psi)\cdot\mathbf{e}_\phi. \tag{48}$$

References

[1] L.C Steinhauer and A Ishida. Nearby-fluids equilibria I. Formalism and transition to single-fluid magnetohydrodynamics. *Physics of Plasmas*, 13:052513, 2006.